Monotonicity of the polaron energy

Tadahiro Miyao
Department of Mathematics, Hokkaido University,
Sapporo 060-0810, Japan
e-mail: miyao@math.sci.hokudai.ac.jp

Abstract

In condensed matter physics, the polaron has been fascinating subject. It is described by the Hamiltonian of H. Fröhlich. In this paper, the Fröhlich Hamiltonian is investigated from a viewpoint of the self-dual cone analysis proposed in [33]. This point of view clarifies the monotonicity of polaron energy, i.e., denoting the lowest energy of the Fröhlich Hamiltonian with the ultraviolet cutoff Λ by E_Λ, we prove $E_\Lambda > E_{\Lambda'}$ for $\Lambda < \Lambda'$.

1 Introduction

Let us consider an electron in an ionic crystal. Through the Coulomb interaction, the electron polarizes the lattice all around itself. Then if the electron moves, the polarization always comes with it. Hence it is natural to regard the electron and the accompanying distortion field as one object, called the *polaron*. This system is described by the polaron Hamiltonian of H. Fröhlich [14] and of interest in condensed matter physics. As a model for a nonrelativistic particle coupled with the bosonic field, this Hamiltonian was widely studied by not only physicists but also mathematicians. Although quite a number of physical literatures can be found, we refer to [5, 10] as accessible works. In general, there are two approaches to tackle the H. Fröhlich Hamiltonian. One is based on the Feynman's path integral approach [6, 9, 42], and other one is standard operator theoretical methods [17, 22, 26, 40, 41]. Nowadays we already reaped a rich harvest from the polaron Hamiltonian by both approaches. However it is still an attractive subject [2, 11, 12, 13, 18, 19, 27, 31, 34, 43].

If one wishes to treat this model rigorously, what we have to clarify first is mathematical definition of the Hamiltonian. Roughly speaking, the Fröhlich polaron Hamiltonian is defined as the limit of ultraviolet cutoff Hamiltonian. It is easily seen that the Hamiltonian with cutoffs is well-defined mathematically, but it is not so obvious that whether the Hamiltonian without ultraviolet cutoffs is mathematically definable or not. The problem of the removal of ultraviolet cutoffs was succesfully investigated by several authors [7, 15, 16, 22, 35, 40, 41].

Then next problem was the spectral analysis of the Hamiltonian without cutoffs. In [15, 16], J. Fröhlich studied the spectral properties of the Hamiltonian with a fixed total momentum after removal of the cutoff. See also [17, 24, 34, 36]. In these papers, the existence of a ground state was also proven. Moreover the uniqueness of the ground state was studied by applying the Perron- Frobenius theorem.

The polaron is a reasonable example for an application of the Perron-Frobenius theorem. There are several beautiful works on this theorem. Many of these have been developed in order to show the uniqueness of the ground state in nonrelativistic quantum field theory [8, 20, 21, 40]. Of course, the polaron has been a target for this theorem as well, but known as a tough problem because of difficulties comming from the removal of ultraviolet cutoffs. In the previous work [32], the author proved the uniqueness of the ground state for the polaron Hamiltonian without cutoffs. A new operator inequality which will be disscused in later sections played an important role. Especially a kind of an operator monotonicity was essential for the proof. Inspired by this work [32], the author proposed a general framework unifying the Perron-Frobenius theory in the linear algebra and the reflection positivity in the quantum field theory. We call this theory the *self-dual cone analysis* [33]. The self-dual cone analysis partly consists of theory about the operator inequality mentioned above. Applying this analysis, we can explain various physical phenomena with a consistent viewpoint [33]. In this paper, we investigate the H. Fröhlich Hamiltonian by this analysis. We can expect to obtain new insight about the polaron from a viewpoint of the self-dual cone analysis. Hopefully the readers will rediscover charms of the polaron through this attempt.

The paper is organized as follows: In Section 2, we define the Fröhlich Hamiltonians and display our results. In Section 3, we explain our strategy of proof as an abstract theorem. Section 4 deals with the second quantization and Sections 5 and 6 with the proofs of main results. In Appendix A, we review some basic facts from the self-dual analysis. In Appendix B, we show a useful energy inequality.

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2 Main results

In this section, we define several Hamiltonians which are our subjects of study, and display our main results. Mathematical definitions of second quantized operators in the Fock space will be given in §4. If readers are unfamiliar with the second quantization, we recommend them to read §4 first.

2.1 Definition of Hamiltonians with the ultraviolet cutoff

Let $\mathfrak{F}$ be the Fock space over $L^2(\mathbb{R}^3)$. For each $\Lambda > 0$, we define a linear operator H_Λ living in $L^2(\mathbb{R}^3_x) \otimes \mathfrak{F}$ by

$$H_\Lambda = -\frac{1}{2}\Delta_x - \sqrt{\alpha}\lambda_0 \int_{|k|\le\Lambda} \mathrm{d}k \, \frac{1}{|k|} \left[\mathrm{e}^{\mathrm{i}k\cdot x} a(k) + \mathrm{e}^{-\mathrm{i}k\cdot x} a(k)^* \right] + N_\mathrm{f}, \tag{2.1}$$

where Δ_x is the Laplacian on $L^2(\mathbb{R}^3_x)$, $\sqrt{\alpha}$ is the electron-phonon coupling strength and $\lambda_0 = 2^{1/4}(2\pi)^{-1}$. $a(k)$ and $a(k)^*$ are the phonon annihilation- and creation operators with commutation relations

$$[a(k), a(k')^*] = \delta(k - k'), \quad [a(k), a(k')] = 0. \tag{2.2}$$

N_f is the number operator formally expressed as $N_\mathrm{f} = \int_{\mathbb{R}^3} \mathrm{d}k a(k)^* a(k)$. (The complete definitions of these operators will be recalled in §4.) Let us denote the smeared annihilation- and creation operators by $a(f)$ and $a(f)^*$ for $f \in L^2(\mathbb{R}^3)$. These are formally expressed as

$$a(f) = \int_{\mathbb{R}^3} \mathrm{d}k \, \overline{f(k)} a(k), \quad a(f)^* = \int_{\mathbb{R}^3} \mathrm{d}k \, f(k) a(k)^*$$

respectively. Then, by the standard bound

$$\| a(f)^\# (N_\mathrm{f} + \mathbb{1})^{-1/2} \| \le \|f\| \tag{2.3}$$

which comes from (4.9), and Kato-Rellich theorem [37], H_Λ is self-adjoint on $\mathrm{dom}(\Delta_x) \cap \mathrm{dom}(N_\mathrm{f})$ and bounded from below. H_Λ is often called the Hamiltonian with an ultraviolet cutoff Λ.

Let P_tot be the total momentum operator defined by

$$P_\mathrm{tot} = -\mathrm{i}\nabla_x + P_\mathrm{f}. \tag{2.4}$$

Here P_f is the phonon momentum operator given by $P_\mathrm{f} = \int_{\mathbb{R}^3} \mathrm{d}k k a(k)^* a(k)$. Then one can check that P_tot commutes with H_Λ, or the total momentum of the system is conserved. This imples that, under a spectral representation of $P_\mathrm{tot} \simeq \int_{\mathbb{R}^3}^\oplus P \mathrm{d}P$, H_Λ is decomposed as $H_\Lambda \simeq \int_{\mathbb{R}^3}^\oplus H_\Lambda(P) \, \mathrm{d}P$, where $H_\Lambda(P)$ is understood as the Hamiltonian at a fixed total momentum P. (Here the symbol $A \simeq B$ means there exists a unitary operator W such that $A = WBW^*$.) To see this more precisely, let $\mathcal{F}_x$ be the Fourier transformation on $L^2(\mathbb{R}^3_x)$ and let $U = \mathcal{F}_x \mathrm{e}^{\mathrm{i}x\cdot P_\mathrm{f}}$. Then we can actually see this unitary operator U gives a spectral representation of P_tot, namely,

$$U P_\mathrm{tot} U^* = \int_{\mathbb{R}^3}^\oplus P \, \mathrm{d}P. \tag{2.5}$$

Moreover one has

$$U H_\Lambda U^* = \int_{\mathbb{R}^3}^\oplus H_\Lambda(P) \, \mathrm{d}P. \tag{2.6}$$

Each $H_\Lambda(P)$ is concretely expressed as

$$H_\Lambda(P) = \frac{1}{2}(P - P_\mathrm{f})^2 - \sqrt{\alpha}\lambda_0 \int_{|k|\le\Lambda} \mathrm{d}k\, \frac{1}{|k|}[a(k) + a(k)^*] + N_\mathrm{f}. \tag{2.7}$$

Then, by (2.3) and Kato-Rellich theorem, $H_\Lambda(P)$ is self-adjoint on $\mathrm{dom}(P_\mathrm{f}^2) \cap \mathrm{dom}(N_\mathrm{f})$, bounded from below. The self-adjoint operator (2.7) is called the Hamiltonian with an ultraviolet cutoff Λ, at a fixed total momentum P.

2.2 Removal of ultraviolet cutoff

One of difficult problems is removal of ultraviolet cutoff Λ. Namely we would like to take a limit $\Lambda \to \infty$ in (2.1) and (2.7). However, since $1/|k|$ is not square integrable, the interaction terms are not well defined in this limit. Therefore the standard perturbation methods, like Kato-Rellich theorem, can not be applicable to define Hamiltonians without ultraviolet cutoff. In other words, we face a singular perturbation problem. Fortunately this problem was already solved by several authors. Here we just state the results.

Proposition 2.1 *One obtains the following.*

(i) *There exists a unique semibounded self-adjoint operator H so that H_Λ converges to H in strong resolvent sense as $\Lambda \to \infty$.*

(ii) *For each $P \in \mathbb{R}^3$, there exists a unique semibounded self-adjoint operator $H(P)$ so that $H_\Lambda(P)$ converges to $H(P)$ in strong resolvent sense as $\Lambda \to \infty$.*

As to the proof of Proposition 2.1, we refer to [17, 35].

2.3 Results

Let

$$E_\Lambda = \inf \mathrm{spec}(H_\Lambda), \quad E = \inf \mathrm{spec}(H). \tag{2.8}$$

Similarly put

$$E_\Lambda(P) = \inf \mathrm{spec}(H_\Lambda(P)), \quad E(P) = \inf \mathrm{spec}(H(P)). \tag{2.9}$$

Our curiousity leads to the following question:

How do E_Λ and $E_\Lambda(P)$ behave as functions of Λ?

Of course, we already know that

$$\lim_{\Lambda\to\infty} E_\Lambda = E, \quad \lim_{\Lambda\to\infty} E_\Lambda(P) = E(P) \tag{2.10}$$

by Proposition 2.1. Hence our real motive behind the question is that we are not satisfied the results of convergence, and we wish to know more detailed information about the behavior of E_Λ and $E_\Lambda(P)$.

In subsequent sections, we will show the following claims.

Theorem 2.2 *For each $P \in \mathbb{R}^3$, $E_\Lambda(P)$ is monotonically decreasing in Λ.*

If the total momentum P is restricted within a certain region, one can claim a stronger result.

Theorem 2.3 *For each $P \in \mathbb{R}^3$ with $|P| < \sqrt{2}$, $E_\Lambda(P)$ is strictly decreasing in Λ.*

Remark 2.4 Theorems 2.2 and 2.3 suggest the polaron at a fixed total momentum is more stable, the larger we take the ultraviolet cutoff. Hence the polaron is energetically most stable if the ultraviolet cutoff is removed.

Since $E_\Lambda = E_\Lambda(0)$ by (2.6) and the well-known property $E_\Lambda(0) \le E_\Lambda(P)$, one has the following corollary.

Corollary 2.5 *E_Λ is strictly decreasing in Λ.*

The above claims are consequences of large theoretical structure, called the self-daul cone analysis proposed in [33]. To apply self-dual cone analysis, we have to choose a physically proper self-dual cone in the Hilbert space. Here the word "physically proper" means we must make a selection of a self-dual cone such that the interaction term of the Hamiltonian becomes attractive. (The true meaning of this sentence will be clarified in the following sections.) If the interaction is attractive under a specific self-dual cone, one can apply various operator theoretical methods in the self-dual cone analysis. Hence choice of a self-dual cone is the heart of our study. In this paper, we will take the *Fröhlich cone* discovered by J. Fröhlich [15, 16]. As we will see, under this choice, the electron-phonon interaction term becomes attractive.

3 Monotonically decreasing self-adjoint operators

In this section, we will provide a strategy of the proof of Theorem 2.2 as an abstract theorem. Indeed readers easily notice a smilarity between Theorem 2.2 and Theorem 3.2. Through our arguments, it is revealed that essential point of the proof of Theorem 2.2 is the operator monotonicity expressed as (3.2).

3.1 Basic definitions

Let $\mathfrak{h}$ be a complex Hilbert space and $\mathfrak{p}$ be a convex cone in $\mathfrak{h}$. Then $\mathfrak{p}$ is called to be *self-dual* if

$$\mathfrak{p} = \{x \in \mathfrak{h} \,|\, \langle x, y\rangle \ge 0 \ \forall y \in \mathfrak{p}\}. \tag{3.1}$$

A typical example of self-dual cone is the standard positive cone in $L^2(\mathbb{R}^d)$ given by $L^2(\mathbb{R}^d)_+ = \{f \in L^2(\mathbb{R}^d) \,|\, f(x) \ge 0 \text{ a.e. } x\}$. Henceforth $\mathfrak{p}$ always denotes the self-dual cone in $\mathfrak{h}$. The following properties of $\mathfrak{p}$ are well-known [4, 23]:

Proposition 3.1 *One has the following.*

(i) $\mathfrak{p} \cap (-\mathfrak{p}) = \{0\}$.

(ii) *There exists a unique involution j in $\mathfrak{h}$ such that $jx = x$ for all $x \in \mathfrak{p}$.*

(iii) *Each element $x \in \mathfrak{h}$ with $jx = x$ has a unique decomposition $x = x_+ - x_-$, where $x_+, x_- \in \mathfrak{p}$ and $\langle x_+, x_- \rangle = 0$.*

(iv) *$\mathfrak{h}$ is linearly spanned by $\mathfrak{p}$.*

If $x - y \in \mathfrak{p}$, then we will write $x \geq y$ (or $y \leq x$) w.r.t. $\mathfrak{p}$. Let A and B be densely defined linear operators on $\mathfrak{h}$. If $Ax \geq Bx$ w.r.t. $\mathfrak{p}$ for all $x \in \mathrm{dom}(A) \cap \mathrm{dom}(B) \cap \mathfrak{p}$, then we will write $A \trianglerighteq B$ (or $B \trianglelefteq A$) w.r.t. $\mathfrak{p}$. Especially if A satisfies $0 \trianglelefteq A$ w.r.t. $\mathfrak{p}$, then we say that A *preserves positivity with respect to* $\mathfrak{p}$. We remark that this symbol "$\trianglerighteq$" was first introduced by Miura [25, 30].

An element x in $\mathfrak{p}$ is called to be *strictly positive* if $\langle x, y \rangle \geq 0$ for all $y \in \mathfrak{p} \backslash \{0\}$. We will write this as $x > 0$ w.r.t. $\mathfrak{p}$. Of course, an inequality $x > y$ w.r.t. $\mathfrak{p}$ means $x - y$ is strictly positive w.r.t. $\mathfrak{p}$. If bounded operators A and B satsify $Ax > Bx$ w.r.t. $\mathfrak{p}$ for all $x \in \mathfrak{p} \backslash \{0\}$, then we will express this as $A \triangleright B$ (or $B \triangleleft A$) w.r.t. $\mathfrak{p}$. Clearly if $A \triangleright B$ w.r.t. $\mathfrak{p}$, then $A \trianglerighteq B$ w.r.t. $\mathfrak{p}$. We say that A *improves positivity w.r.t.* $\mathfrak{p}$ if $A \triangleright 0$ w.r.t. $\mathfrak{p}$.

Fundamental natures of these inequalities are reviewed in Appendix A.

3.2 Abstract theorem

Let $\mathfrak{p}$ be a self-dual cone in the Hilbert space $\mathfrak{h}$. Let $\{H_n\}_{n \in \mathbb{N}}$ be a sequence of semibounded self-adjoint operators on $\mathfrak{h}$. In this subsection we always assume the following.

(A. 1) There exists a unique semibounded self-adjoint operator H such that H_n converges to H in strong resolvent sense.

(A. 2) Each H_n has a common domain $\mathscr{D}$.

(A. 3) For all $n \in \mathbb{N}$ and $s \geq 0$, $\mathrm{e}^{-sH_n} \trianglerighteq 0$ w.r.t. $\mathfrak{p}$.

Theorem 3.2 *Assume (A. 1), (A. 2), (A. 3). In addition assume*

$$H_1 \trianglerighteq H_2 \trianglerighteq \cdots \trianglerighteq H_n \trianglerighteq H_{n+1} \trianglerighteq \cdots \qquad \textit{w.r.t. } \mathfrak{p}. \tag{3.2}$$

Set $E_n = \inf \mathrm{spec}(H_n)$ and $E = \inf \mathrm{spec}(H)$. Then $\{E_n\}_n$ is monotonically decreasing in n:

$$E_1 \geq E_2 \geq \cdots \geq E_n \geq E_{n+1} \geq \cdots \geq E \tag{3.3}$$

and $E = \lim_{n \to \infty} E_n$.

3.3 Proof of Theorem 3.2

Let j be the involution in Proposition 3.1. Since $\mathrm{e}^{-tH_n} \trianglerighteq 0$ w.r.t. $\mathfrak{p}$ for all $n \in \mathbb{N}$, H_n must be j-real in the sense that $H_n j = jH_n$. From this fact, it follows

$$E_n = \inf \Big\{ \langle \varphi, H_n \varphi \rangle \, \Big| \, \varphi \in \mathscr{D}, \ j\varphi = \varphi, \ \|\varphi\| = 1 \Big\}. \tag{3.4}$$

Indeed we observe, since H_n is j-real,

$$\langle \varphi, H_n \varphi \rangle = \big\langle \Re\varphi, H_n \Re\varphi \big\rangle + \big\langle \Im\varphi, H_n \Im\varphi \big\rangle, \tag{3.5}$$

where $\Re\varphi = \frac{1}{2}(\mathbb{1} + j)\varphi$, $\Im\varphi = \frac{1}{2\mathrm{i}}(\mathbb{1} - j)\varphi$. Clearly $j\Re\varphi = \Re\varphi$, $j\Im\varphi = \Im\varphi$ and $\|\varphi\|^2 = \|\Re\varphi\|^2 + \|\Im\varphi\|^2$. Hence

$$E_n \geq \text{RHS of (3.4)} \tag{3.6}$$

holds. The converse inequality is trivial.

Fix $\varepsilon > 0$ arbitrarily. By (3.4), we can choose $\varphi \in \mathscr{D}$ so that $j\varphi = \varphi$ and $\langle \varphi, H_n \varphi \rangle \leq E_n + \varepsilon$. Remark that, by Proposition 3.1 (iii), we can express φ as $\varphi = \varphi_+ - \varphi_-$ so that $\varphi_\pm \in \mathfrak{p}$. Define $|\varphi|_\mathfrak{p} = \varphi_+ + \varphi_-$. Applying Theorem A.5, one obtains both $\langle |\varphi|_\mathfrak{p}, H_n |\varphi|_\mathfrak{p} \rangle$ and $\langle |\varphi|_\mathfrak{p}, H_{n+1} |\varphi|_\mathfrak{p} \rangle$ are finite, and

$$\begin{aligned} \langle \varphi, H_n \varphi \rangle &\geq \langle |\varphi|_\mathfrak{p}, H_n |\varphi|_\mathfrak{p} \rangle \\ &= \langle |\varphi|_\mathfrak{p}, H_{n+1} |\varphi|_\mathfrak{p} \rangle + \langle |\varphi|_\mathfrak{p}, (H_n - H_{n+1}) |\varphi|_\mathfrak{p} \rangle. \end{aligned} \tag{3.7}$$

By the monotonicity (3.2), $\langle |\varphi|_\mathfrak{p}, (H_n - H_{n+1}) |\varphi|_\mathfrak{p} \rangle \geq 0$ holds. Now we arrive at

$$E_n + \varepsilon \geq \langle |\varphi|_\mathfrak{p}, H_{n+1} |\varphi|_\mathfrak{p} \rangle \geq E_{n+1}. \tag{3.8}$$

Since $\varepsilon > 0$ is arbitrary, one obtains the assertion. $\square$

4 Second quantization

4.1 Basic definitions

The bosonic Fock space over $\mathfrak{h}$ is defined by

$$\mathfrak{F}(\mathfrak{h}) = \sum_{n \geq 0}^{\oplus} \mathfrak{h}^{\otimes_\mathrm{s} n}, \tag{4.1}$$

where $\mathfrak{h}^{\otimes_\mathrm{s} n}$ is the n-fold symmetric tensor product of $\mathfrak{h}$ with convention $\mathfrak{h}^{\otimes_\mathrm{s} 0} = \mathbb{C}$. The vector $\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathfrak{F}(\mathfrak{h})$ is called the Fock vacuum. For each $n \in \{0\} \cup \mathbb{N}$, let P_n be an orthogonal projection defined by $P_n \varphi = \sum_{n \geq j \geq 0}^{\oplus} \varphi_j$ for all $\varphi = \sum_{j \geq 0}^{\oplus} \varphi_j \in \mathfrak{F}$. Then an important dense subspace of $\mathfrak{F}(\mathfrak{h})$, called the finite particle subspace, is defined by

$$\mathfrak{F}_{\mathrm{fin}}(\mathfrak{h}) = \bigcup_{n \geq 0} P_n \mathfrak{F}(\mathfrak{h}). \tag{4.2}$$

We denote by $a(f)$ ($f \in \mathfrak{h}$) the annihilation operator on $\mathfrak{F}(\mathfrak{h})$, its adjoint $a(f)^*$, called the creation operator, is defined by

$$a(f)^*\varphi = \sum_{n\geq 1}^{\oplus} \sqrt{n} S_n(f \otimes \varphi^{(n-1)}) \tag{4.3}$$

for $\varphi = \sum_{n\geq 0}^{\oplus} \varphi^{(n)} \in \mathrm{dom}(a(f)^*)$, where S_n is the symmetrizer on $\mathfrak{h}^{\otimes_s n}$. It is well-known that the annihilation- and creation operators satisfy the canonical commutation relations or CCRs

$$[a(f), a(g)^*] = \langle f, g\rangle, \quad [a(f), a(g)] = 0 = [a(f)^*, a(g)^*] \tag{4.4}$$

on $\mathfrak{F}_{\mathrm{fin}}$.

Let C be a contraction operator on $\mathfrak{h}$, that is , $\|C\| \leq 1$. Then we define a contraction operator $\Gamma(C)$ on $\mathfrak{F}(\mathfrak{h})$ by

$$\Gamma(C) = \sum_{n\geq 0}^{\oplus} C^{\otimes n} \tag{4.5}$$

with $C^{\otimes 0} = \mathbb{1}$, the identity operator. For a self-adjoint operator A on $\mathfrak{h}$, let us introduce

$$\mathrm{d}\Gamma(A) = 0 \oplus \sum_{n\geq 1}^{\oplus} \sum_{n\geq k\geq 1} \mathbb{1}^{\otimes(k-1)} \otimes A \otimes \mathbb{1}^{\otimes(n-k)} \tag{4.6}$$

acting in $\mathfrak{F}(\mathfrak{h})$. Then $\mathrm{d}\Gamma(A)$ is essentially self-adjoint. We denote its closure by the same symbol. A typical example is the boson number operator $N_{\mathrm{f}} = \mathrm{d}\Gamma(\mathbb{1})$. We remark the following relation between $\Gamma(\cdot)$ and $\mathrm{d}\Gamma(\cdot)$:

$$\Gamma(\mathrm{e}^{\mathrm{i}tA}) = \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(A)}. \tag{4.7}$$

In particular if A is positive, then one has

$$\Gamma(\mathrm{e}^{-tA}) = \mathrm{e}^{-t\mathrm{d}\Gamma(A)}. \tag{4.8}$$

Let A be a positive self-adjoint operator. Then the following operator inequalities are well-known:

$$a(f)^*a(f) \leq \|A^{-1/2}f\|^2(\mathrm{d}\Gamma(A) + \mathbb{1}), \tag{4.9}$$

$$\mathrm{d}\Gamma(A) + a(f) + a(f)^* \geq -\|A^{-1/2}f\|^2. \tag{4.10}$$

4.2 Fock space over L^2-space

In this paper, the bosonic Fock space over $L^2(\mathbb{R}^3_k) = L^2(\mathbb{R}^3, \mathrm{d}k)$ will often appear and we simply denote as

$$\mathfrak{F} = \mathfrak{F}(L^2(\mathbb{R}^3_k)). \tag{4.11}$$

Also the corresponding finite particle subspace $\mathfrak{F}_{\mathrm{fin}}(L^2(\mathbb{R}^3_k))$ is denoted by $\mathfrak{F}_{\mathrm{fin}}$. The n-boson subspace $L^2(\mathbb{R}^3_k)^{\otimes_s n}$ is naturally identified with $L^2_{\mathrm{sym}}(\mathbb{R}^{3n}) = \big\{\varphi \in$

$L^2(\mathbb{R}_k^{3n}) \,|\, \varphi(k_1,\ldots,k_n) = \varphi(k_{\sigma(1)},\ldots,k_{\sigma(n)})$ a.e. $\forall \sigma \in \mathfrak{S}_n \big\}$, where $\mathfrak{S}_n$ is the permutation group on a set $\{1,2,\ldots,n\}$. Hence

$$\mathfrak{F} = \mathbb{C} \oplus \sum_{n\ge 1}^{\oplus} L^2_{\mathrm{sym}}(\mathbb{R}_k^{3n}). \tag{4.12}$$

The annihilation- and creation operators are symbolically expressed as

$$a(f) = \int_{\mathbb{R}^3} \mathrm{d}k\, \overline{f(k)} a(k), \quad a(f)^* = \int_{\mathbb{R}^3} \mathrm{d}k\, f(k) a(k)^*. \tag{4.13}$$

If ω is a multipilication operator by the function $\omega(k)$, then $\mathrm{d}\Gamma(\omega)$ is formally written as

$$\mathrm{d}\Gamma(\omega) = \int_{\mathbb{R}_k^3} \mathrm{d}k\, \omega(k) a(k)^* a(k). \tag{4.14}$$

4.3 The Fröhlich cone in $\mathfrak{F}$

In order to discuss the inequalities introduced in §3, we have to determine a self-dual cone in $\mathfrak{F}$. Here we will introduce a natural self-dual cone in $\mathfrak{F}$ which is suitable for our analysis in later sections.

Set

$$\mathfrak{F}_+^{(n)} = \big\{\varphi \in L^2(\mathbb{R}_k^3)^{\otimes_{\mathrm{s}} n} \,|\, \langle f_1 \otimes \cdots \otimes f_n, \varphi\rangle \ge 0 \;\forall f_1,\ldots,\forall f_n \in L^2(\mathbb{R}_k^3)_+ \big\} \tag{4.15}$$

with $\mathfrak{F}_+^{(0)} = \mathbb{R}_+ = \{r \in \mathbb{R} \,|\, r \ge 0\}$. Then each $\mathfrak{F}_+^{(n)}$ is a self-dual cone in $L^2(\mathbb{R}_k^3)^{\otimes_{\mathrm{s}} n}$. Under the natural identification $L^2(\mathbb{R}_k^3)^{\otimes_{\mathrm{s}} n} = L^2_{\mathrm{sym}}(\mathbb{R}_k^{3n})$, one sees

$$\mathfrak{F}_+^{(n)} = \big\{\varphi \in L^2_{\mathrm{sym}}(\mathbb{R}^{3n}) \,|\, \varphi(k_1,\ldots,k_n) \ge 0 \text{ a.e.}\big\}. \tag{4.16}$$

Now we define

$$\mathfrak{F}_+ = \sum_{n\ge 0}^{\oplus} \mathfrak{F}_+^{(n)}. \tag{4.17}$$

Again $\mathfrak{F}_+$ becomes to be a self-dual cone in $\mathfrak{F}$.

Definition 4.1 $\mathfrak{F}_+$ is referred to as the *Fröhlich cone.*

Remark 4.2 The Fröhlich cone was introduced by Fröhlich [15, 16] to study the quantum field theory.

We summarize properties of operators in $\mathfrak{F}$ below.

Proposition 4.3 *Let C be a contraction on $L^2(\mathbb{R}_k^3)$. Then if $C \trianglerighteq 0$ w.r.t. $L^2(\mathbb{R}_k^3)_+$, one has $\Gamma(C) \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$. Especially one has the following.*

(i) *For a self-adjoint operator A, if $\mathrm{e}^{\mathrm{i}tA} \trianglerighteq 0$ w.r.t. $L^2(\mathbb{R}^3)_+$, then one has $\Gamma(\mathrm{e}^{\mathrm{i}tA}) \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$.*

(ii) *For a positive self-adjoint operator B, if* $\mathrm{e}^{-tB} \trianglerighteq 0$ *w.r.t.* $L^2(\mathbb{R}^3)_+$*, then one has* $\Gamma(\mathrm{e}^{-tB}) \trianglerighteq 0$ *w.r.t.* $\mathfrak{F}_+$.

Proof. For each $f_1, \ldots, f_n \in L^2(\mathbb{R}^3)_+$ and $\varphi \in \mathfrak{F}_+$, one can check that

$$\langle \Gamma(C)\varphi, f_1 \otimes \cdots \otimes f_n \rangle = \langle \varphi, Cf_1 \otimes \cdots \otimes Cf_n \rangle \geq 0. \tag{4.18}$$

This means $\Gamma(C) \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$. □

Proposition 4.4 *If* $f \geq 0$ *w.r.t.* $L^2(\mathbb{R}^3_k)_+$*, then* $a(f)^* \trianglerighteq 0$ *and* $a(f) \trianglerighteq 0$ *w.r.t.* $\mathfrak{F}_+$.

Proof. By (4.3), for any $g_1, \ldots, g_n \in L^2(\mathbb{R}^3_k)_+$ and $\varphi \in \mathfrak{F}_+ \cap \mathrm{dom}(a(f)^*)$, one has

$$\langle a(f)^*\varphi, g_1 \otimes \cdots \otimes g_n \rangle = \sqrt{n}\langle f \otimes \varphi^{(n-1)}, S_n g_1 \otimes \cdots \otimes g_n \rangle \geq 0. \tag{4.19}$$

This implies $a(f)^* \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$. □

Proposition 4.5 (Ergodicity) *For each* $f \in L^2(\mathbb{R}^3_k)$*, let* $\phi(f)$ *be a linear operator defined by*

$$\phi(f) = a(f) + a(f)^*. \tag{4.20}$$

If $f > 0$ *w.r.t.* $L^2(\mathbb{R}^3_k)_+$*, that is,* $f(k) > 0$ *a.e.* k*, then* $\phi(f)$ *is ergodic in the sense that, for any* $x, y \in (\mathfrak{F}_+ \cap \mathfrak{F}_{\mathrm{fin}}) \backslash \{0\}$*, there exists an* $n \in \{0\} \cup \mathbb{N}$ *such that* $\langle x, \phi(f)^n y \rangle > 0$.

Proof. First we remark that if $f \geq 0$ w.r.t. $L^2(\mathbb{R}^3_k)_+$, $\phi(f) \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$ by Proposition 4.4. Moreover $\mathfrak{F}_{\mathrm{fin}} \subseteq \mathrm{dom}(\phi(f)^n)$ for any $n \in \mathbb{N}$.

Write $x, y \in \mathfrak{F}_+ \backslash \{0\}$ as $x = \sum_{n\geq 0}^{\oplus} x^{(n)}$ and $y = \sum_{n\geq 0}^{\oplus} y^{(n)}$. Each $x^{(n)}$ and $y^{(n)}$ are in $\mathfrak{F}_+^{(n)}$. Since both x and y are nonzero vecors in $\mathfrak{F}_+$, there exist $p, q \in \{0\} \cup \mathbb{N}$ so that $x^{(p)} \in \mathfrak{F}_+^{(p)}$ and $y^{(q)} \in \mathfrak{F}_+^{(q)}$. Clearly $x \geq \sum_{n\geq 0}^{\oplus} \delta_{np} x^{(n)}$ w.r.t. $\mathfrak{F}_+$ and $y \geq \sum_{n\geq 0}^{\oplus} \delta_{nq} y^{(n)}$ w.r.t. $\mathfrak{F}_+$, where δ_{mn} is Kronecker delta. Hence one has

$$\langle x, \phi(f)^{p+q} y \rangle \geq \langle x^{(p)}, \phi(f)^{p+q} y^{(q)} \rangle. \tag{4.21}$$

Set $\phi_-(f) = a(f)$ and $\phi_+(f) = a(f)^*$. Then since $\phi_\pm(f) \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$ provided that $f \in L^2(\mathbb{R}^3_k)_+$, we have $\phi(f)^{p+q} \trianglerighteq \phi_+(f)^p \phi_-(f)^q$ w.r.t. $\mathfrak{F}_+$. Accordingly one has

$$\phi(f)^p x^{(p)} \geq \phi_-(f)^p x^{(p)} = \sqrt{p!}\langle f^{\otimes p}, x^{(p)} \rangle \Omega, \tag{4.22}$$

$$\phi(f)^q y^{(q)} \geq \phi_-(f)^q y^{(q)} = \sqrt{q!}\langle f^{\otimes q}, y^{(q)} \rangle \Omega \tag{4.23}$$

w.r.t. $\mathfrak{F}_+$. By the assumption $f > 0$ w.r.t. $L^2(\mathbb{R}^3_k)_+$, $\langle f^{\otimes p}, x^{(p)} \rangle > 0$ and $\langle f^{\otimes q}, y^{(q)} \rangle > 0$ hold. Hence we arrive at, by (4.21),

$$\langle x, \phi(f)^{p+q} y \rangle \geq \sqrt{p!q!}\langle f^{\otimes p}, x^{(p)} \rangle \langle f^{\otimes q}, y^{(q)} \rangle > 0. \tag{4.24}$$

This proves the assertion. □

4.4 Local properties

Let B_Λ be a ball of radius Λ in $\mathbb{R}^3_k$ and let χ_Λ be a function on $\mathbb{R}^3$ defined by $\chi_\Lambda(k) = 1$ if $k \in B_\Lambda$ and $\chi_\Lambda(k) = 0$ otherwise. Then as a multiplication operator, χ_Λ is an orthogonal projection on $L^2(\mathbb{R}^3_k)$ and $Q_\Lambda = \Gamma(\chi_\Lambda)$ is also an orthogonal projection on $\mathfrak{F}$. Now let us define the local Fock space by

$$\mathfrak{F}_\Lambda = Q_\Lambda \mathfrak{F}. \tag{4.25}$$

Clearly $\mathfrak{F} = \mathfrak{F}_\infty$. Since $\chi_\Lambda L^2(\mathbb{R}^3_k) = L^2(B_\Lambda)$, $\mathfrak{F}_\Lambda$ can be identified with $\mathfrak{F}(L^2(B_\Lambda))$. As to the annihilation- and creation operators, we remark the following properties:

$$a(f)Q_\Lambda = a(\chi_\Lambda f) = \int_{|k|\le\Lambda} \mathrm{d}k\, \overline{f(k)} a(k), \tag{4.26}$$

$$Q_\Lambda a(f)^* = a(\chi_\Lambda f)^* = \int_{|k|\le\Lambda} \mathrm{d}k\, f(k) a(k)^*, \tag{4.27}$$

$$\mathrm{d}\Gamma(\omega)Q_\Lambda = \mathrm{d}\Gamma(\chi_\Lambda \omega) = \int_{|k|\le\Lambda} \mathrm{d}k\, \omega(k) a(k)^* a(k). \tag{4.28}$$

A natural self-dual cone in $\mathfrak{F}_\Lambda$ would be the following one. First let us define

$$\mathfrak{F}^{(n)}_{\Lambda,+} = \left\{ \varphi \in L^2(B_\Lambda)^{\otimes_s n} \,|\, \langle f_1 \otimes \cdots \otimes f_n, \varphi \rangle \ge 0 \ \forall f_1, \dots, \forall f_n \in L^2(B_\Lambda)_+ \right\} \tag{4.29}$$

with $\mathfrak{F}^{(0)}_{\Lambda,+} = \mathbb{R}^+$, where $L^2(B_\Lambda)_+ = \{ f \in L^2(B_\Lambda) \,|\, f(k) \ge 0 \text{ a.e. } \}$. Then we introduce

$$\mathfrak{F}_{\Lambda,+} = \sum\nolimits^{\oplus}_{n\ge 0} \mathfrak{F}^{(n)}_{\Lambda,+}. \tag{4.30}$$

$\mathfrak{F}_{\Lambda,+}$ is a self-dual cone in $\mathfrak{F}_\Lambda$.

Definition 4.6 $\mathfrak{F}_{\Lambda,+}$ is referred to as the *local Fröhlich cone.*

Proposition 4.7 *Propositions 4.3, 4.4 and 4.5 are still true even if one replaces $L^2(\mathbb{R}^3_k)_+$ and $\mathfrak{F}_+$ by $L^2(B_\Lambda)_+$ and $\mathfrak{F}_{\Lambda,+}$ respectively.*

5 Proof of Theorem 2.2

We will prove the theorem by series of propositions. Throughout our arguments, it will be clarified the self-dual cone analysis plays important roles.

5.1 Reduction

Our strategy of the proof of Theorem 2.2 is simple: we just apply Theorem 3.2. Thus what we have to do is to check every assumptions in Theorem 3.2.

The assumption (A. 1) is satisfied by Proposition 2.1. (A. 2) is satisfied as well, because $\mathrm{dom}(H_\Lambda(P)) = \mathrm{dom}(N_\mathrm{f}) \cap \mathrm{dom}(P_\mathrm{f}^2)$ for each $\Lambda > 0$. This is an immediate consequence of the Kato-Rellich theorem. Therefore it suffices to show the following two propositions.

Proposition 5.1 *For all $P \in \mathbb{R}^3$, $\Lambda > 0$ and $s \geq 0$, we have $\mathrm{e}^{-sH_\Lambda(P)} \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$.*

The above proposition corresponds to the assumption (A. 3). Next propositon means the assumption (3.2) is fulfilled.

Proposition 5.2 *For each $P \in \mathbb{R}^3$, $\{H_\Lambda(P)\}_\Lambda$ is monotonically decreasing in a sense that if $\Lambda \leq \Lambda'$, then $H_\Lambda(P) \trianglerighteq H_{\Lambda'}(P)$ w.r.t. $\mathfrak{F}_+$.*

In the remainder of this section, we will show two propositions above.

5.2 Proof of Proposition 5.1

Throughout this paper, we avoid a style of giving the shortest proof, but we will prove our claims with emphasizing physical meanings from viewpoint of the self-dual cone analysis.

Let us write the Hamiltonian $H_\Lambda(P)$ as

$$H_\Lambda(P) = H_0(P) - V_\Lambda, \tag{5.1}$$

where

$$H_0(P) = \frac{1}{2}(P - P_\mathrm{f})^2 + N_\mathrm{f}, \tag{5.2}$$

$$V_\Lambda = \sqrt{\alpha}\lambda_0 \int_{|k|\leq\Lambda} \mathrm{d}k \, \frac{1}{|k|}[a(k) + a(k)^*]. \tag{5.3}$$

Clearly V_Λ is the electron-phonon interaction term.

Lemma 5.3 *For all $P \in \mathbb{R}^3$ and $\Lambda > 0$, one obtains the following.*

(i) *$\mathrm{e}^{-tH_0(P)} \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$ for all $t \geq 0$.*

(ii) *(Attraction) $-V_\Lambda$ is attractive w.r.t. $\mathfrak{F}_+$ in a sense $-V_\Lambda \trianglelefteq 0$ w.r.t. $\mathfrak{F}_+$.*

Proof. (i) By Proposition 4.3, $\mathrm{e}^{-tN_\mathrm{f}} \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$. Furthermore $\mathrm{e}^{-t(P-P_\mathrm{f})^2} \trianglerighteq 0$ w.r.t. $\mathfrak{F}$ for all P. [Proof: We can write $\mathrm{e}^{-t(P-P_\mathrm{f})^2} = \mathrm{e}^{-t|P|^2} \oplus \sum_{n\geq1}^{\oplus} \exp\{-t(P - \sum_{j=1}^n k_j)^2\}$. Each n-th component satisfies $\exp\{-t(P - \sum_{j=1}^n k_j)^2\} \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+^{(n)}$.] This implies $\exp[-tH_0(P)] = \exp[-t\frac{1}{2}(P - P_\mathrm{f})^2]\exp[-tN_\mathrm{f}] \trianglerighteq 0$ w.r.t. $\mathfrak{F}_+$ for all P.

(ii) This immediately follows from Proposition 4.4. □

Now we are ready to prove Proposition 5.1. By Lemma 5.3, every assumptions in Corollary A.4 are proven already. Thus Proposition 5.1 follows from Corollary A.4. □

5.3 Proof of Propositon 5.2

First of all, we will clarify a property of V_Λ.

Lemma 5.4 *$-V_\Lambda$ is monotonically decreasing in Λ in a sense $-V_\Lambda \trianglerighteq -V_{\Lambda'}$ w.r.t. $\mathfrak{F}_+$ provided $\Lambda \le \Lambda'$.*

Remark 5.5 By Lemma 5.4, the electron-phonon interaction becomes stronger, the larger we take the ultraviolet cutoff. (Here our choice of the Fröhlich cone is essential.) This viewpoint is characteristic of the self-dual cone analysis.

Proof. Remark that, since $\mathrm{dom}(V_\Lambda)\cap\mathrm{dom}(V_{\Lambda'}) \supseteq \mathfrak{F}_{\mathrm{fin}}$, we see $\mathrm{dom}(V_\Lambda)\cap\mathrm{dom}(V_{\Lambda'})\cap \mathfrak{F}_+ \neq \{0\}$. Choose $\varphi \in \mathrm{dom}(V_\Lambda)\cap\mathrm{dom}(V_{\Lambda'})\cap\mathfrak{F}_+$. Then, applying Propositon 4.4, we have

$$(V_{\Lambda'} - V_\Lambda)\varphi = \sqrt{\alpha}\lambda_0 \int_{\Lambda<|k|\le\Lambda'} \mathrm{d}k\, \frac{1}{|k|}[a(k) + a(k)^*]\varphi \ge 0 \tag{5.4}$$

w.r.t. $\mathfrak{F}_+$. This means $V_{\Lambda'} \trianglerighteq V_\Lambda$ w.r.t. $\mathfrak{F}_+$. □

Assume $\Lambda' \ge \Lambda$. For $\varphi \in \mathrm{dom}(N_{\mathrm{f}}) \cap \mathrm{dom}(P_{\mathrm{f}}^2) \cap \mathfrak{F}_+$, observe, by Lemma 5.4,

$$\big(H_\Lambda(P) - H_{\Lambda'}(P)\big)\varphi = (V_{\Lambda'} - V_\Lambda)\varphi \ge 0 \tag{5.5}$$

w.r.t. $\mathfrak{F}_+$. Since $\mathrm{dom}(N_{\mathrm{f}})\cap\mathrm{dom}(P_{\mathrm{f}}^2)$ is the common domain of $H_\Lambda(P)$ and $H_{\Lambda'}(P)$, we conclude $H_\Lambda(P) \trianglerighteq H_{\Lambda'}(P)$ w.r.t. $\mathfrak{F}_+$ for all $P \in \mathbb{R}^3$. This proves Proposition 5.2. □

6 Proof of Theorem 2.3

6.1 Local Hamiltonian

By the factorization $\mathfrak{F}(\mathfrak{h}_0 \oplus \mathfrak{h}_1) = \mathfrak{F}(\mathfrak{h}_0) \otimes \mathfrak{F}(\mathfrak{h}_1)$, one has

$$\begin{aligned}\mathfrak{F} =& \mathfrak{F}(L^2(B_\Lambda) \oplus L^2(B_\Lambda^{\mathrm{c}})) = \mathfrak{F}(L^2(B_\Lambda)) \otimes \mathfrak{F}(L^2(B_\Lambda^{\mathrm{c}}))\\ =& \sum\nolimits_{n\ge0}^{\oplus} \mathfrak{F}_\Lambda \otimes L^2_{\mathrm{sym}}(B_\Lambda^{\mathrm{c}\times n}) = \mathfrak{F}_\Lambda \oplus \sum\nolimits_{n\ge1}^{\oplus} L^2_{\mathrm{sym}}(B_\Lambda^{\mathrm{c}\times n}; \mathfrak{F}_\Lambda),\end{aligned} \tag{6.6}$$

where $L^2_{\mathrm{sym}}(B_\Lambda^{\mathrm{c}\times n}; \mathfrak{F}_\Lambda)$ is the space of symmetric square integrable $\mathfrak{F}_\Lambda$-valued functions on $B_\Lambda^{\mathrm{c}\times n}$ and $B_\Lambda^{\mathrm{c}} = \mathbb{R}^3\backslash B_\Lambda$. Under this identification, we see that

$$H_\Lambda(P) = K_\Lambda(P) \oplus \sum\nolimits_{n\ge1}^{\oplus} \int_{B_\Lambda^{\mathrm{c}\times n}}^{\oplus} \Big[K_\Lambda(P - k_1 - \cdots - k_n) + n\Big]\, \mathrm{d}k_1\cdots\mathrm{d}k_n. \tag{6.7}$$

Here $K_\Lambda(P)$ is the *local Hamiltonian* defined by

$$K_\Lambda(P) = \frac{1}{2}(P - P_{\mathrm{f},\Lambda})^2 + N_{\mathrm{f},\Lambda} - V_\Lambda, \tag{6.8}$$

where

$$P_{\mathrm{f},\Lambda} = \int_{|k|\leq\Lambda} \mathrm{d}k\, k a(k)^* a(k), \quad N_{\mathrm{f},\Lambda} = \int_{|k|\leq\Lambda} \mathrm{d}k\, a(k)^* a(k). \tag{6.9}$$

$K_\Lambda(P)$ lives in the local Fock space $\mathfrak{F}_\Lambda$. By the Kato-Rellich theorem, it is self-adjoint on $\mathrm{dom}(P_{\mathrm{f},\Lambda}^2) \cap \mathrm{dom}(N_{\mathrm{f},\Lambda})$.

Put

$$L(P) = \frac{1}{2}(P - P_{\mathrm{f},\Lambda})^2 + N_{\mathrm{f},\Lambda}.$$

Obviously $K_\Lambda(P) = L(P) - V_\Lambda$.

Lemma 6.1 *For each $\Lambda > 0$ and $P \in \mathbb{R}^3$, one has the following.*

(i) $\mathrm{e}^{-tL(P)} \trianglerighteq 0$ *w.r.t.* $\mathfrak{F}_{\Lambda,+}$ *for all* $t \geq 0$.

(ii) *(Attraction)* $-V_\Lambda$ *is attractive w.r.t.* $\mathfrak{F}_{\Lambda,+}$ *in a sense that* $-V_\Lambda \trianglelefteq 0$ *w.r.t.* $\mathfrak{F}_{\Lambda,+}$.

Proof. This can be proven in a similar way in the proof of Lemma 5.3. □

Corollary 6.2 *For each $\Lambda > 0$ and $P \in \mathbb{R}^3$, one obtains*

$$\mathrm{e}^{-tK_\Lambda(P)} \trianglerighteq 0 \tag{6.10}$$

w.r.t. $\mathfrak{F}_{\Lambda,+}$ *for all* $t \geq 0$.

Proof. Apply Corollary A.4. □

As to $\mathrm{e}^{-tK_\Lambda(P)}$, we can show a stronger result as follow.

Proposition 6.3 *For any $\Lambda > 0, P \in \mathbb{R}^3$ and $t > 0$, one obtains*

$$\mathrm{e}^{-tK_\Lambda(P)} \rhd 0 \tag{6.11}$$

w.r.t. $\mathfrak{F}_{\Lambda,+}$.

Proof. Essential idea is comming from [15, 16]. Set $F(k) = \sqrt{\alpha}\lambda_0\chi_\Lambda(k)/|k|$. Since $F > 0$ w.r.t. $L^2(B_\Lambda)_+$, V_Λ is ergodic by Proposition 4.7.

By the Duhamel formula, one observes

$$\mathrm{e}^{-tK_\Lambda(P)} = \sum_{j\geq 0} D_j(t) \tag{6.12}$$

with

$$D_j(t) = \int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2 \cdots \int_0^{t-\sum_{i=1}^{j-1} s_i} \mathrm{d}s_j$$
$$\mathrm{e}^{-s_1 L(P)} V_\Lambda \mathrm{e}^{-s_2 L(P)} \cdots \mathrm{e}^{-s_j L(P)} V_\Lambda \mathrm{e}^{-(t-\sum_{i=1}^{j} s_i) L(P)}. \tag{6.13}$$

Since each $D_j(t) \trianglerighteq 0$ w.r.t. $\mathfrak{F}_{\Lambda,+}$ by Lemma 6.1, one has $\mathrm{e}^{-tK_\Lambda(P)} \trianglerighteq D_j(t)$ w.r.t. $\mathfrak{F}_{\Lambda,+}$ for any j. Hence it sufficies to show that a sequence $\{D_j(t)\}_j$ is ergodic in the sense that, for any $\varphi, \psi \in \mathfrak{F}_{\Lambda,+}\backslash\{0\}$, there exists some $N \in \{0\} \cup \mathbb{N}$ such that $\langle \varphi, D_N(t)\psi\rangle > 0$. To this end, write $\varphi = \sum_{n\ge 0}^{\oplus} \varphi^{(n)}$ and $\psi = \sum_{n\ge 0}^{\oplus} \psi^{(n)}$. Then since both φ and ψ are non-zero, there are $p, q \in \{0\}\cup\mathbb{N}$ such that $\varphi^{(p)} \in \mathfrak{F}_{\Lambda,+}^{(p)}\backslash\{0\}$ and $\psi^{(q)} \in \mathfrak{F}_{\Lambda,+}^{(q)}\backslash\{0\}$. Since $\varphi \ge \varphi^{(p)}$ and $\psi \ge \psi^{(q)}$ w.r.t. $\mathfrak{F}_{\Lambda,+}$, one sees

$$\langle \varphi, D_j(t)\psi\rangle \ge \langle \varphi^{(p)}, D_j(t)\psi^{(q)}\rangle \tag{6.14}$$

for any j. Observe that, by the local ergodicity of V_Λ (Proposition 4.7), there exists an $N \in \{0\} \cup \mathbb{N}$, such that $\langle \varphi^{(p)}, V_\Lambda^N \mathrm{e}^{-tL(P)}\psi^{(q)}\rangle > 0$. This implies $\langle \varphi^{(p)}, D_N(t)\psi^{(q)}\rangle > 0$. Hence combining this with (6.14), $\{D_j(t)\}_j$ is ergodic. □

6.2 Proof by the local operator properties

We will prove Theorem 2.3 by clarifying relations between $K_\Lambda(P)$ and $H_\Lambda(P)$.

Lemma 6.4 *Let $K_\Lambda(P)$ be the local Hamiltonian defined by (6.8). Let $\mathcal{E}_\Lambda(P) =$* $\inf \mathrm{spec}(K_\Lambda(P))$. *Then, for $|P| < \sqrt{2}$, one has*

$$\mathcal{E}_\Lambda(P) = E_\Lambda(P). \tag{6.15}$$

Proof. Using the property $\mathcal{E}_\Lambda(0) \le \mathcal{E}_\Lambda(P)$ (Lemma B.1), one has

$$E_\Lambda(P) = \min\{\mathcal{E}_\Lambda(P), \mathcal{E}_\Lambda(0) + 1\} \tag{6.16}$$

by (6.7). On the other hand, we see that

$$\mathcal{E}_\Lambda(P) \le \mathcal{E}_\Lambda(0) + \frac{P^2}{2}. \tag{6.17}$$

[Proof: For each normalized φ, $\langle \varphi, [K_\Lambda(P) - \frac{P^2}{2}]\varphi\rangle$ is linear in P. Hence $F(P) = \mathcal{E}_\Lambda(P) - \frac{P^2}{2}$ is concave. Now we have $F(0) = F(\frac{P}{2} - \frac{P}{2}) \ge \frac{1}{2}F(P) + \frac{1}{2}F(-P)$. Finally using the fact $F(-P) = F(P)$ which can be proven by, for example, the time reversal symmetry [29], we conclude (6.17).] Combining (6.16) and (6.17), we have the assertion. □

By the above lemma, it sufficies to consider the local Hamiltonian $K_\Lambda(P)$ instead of $H_\Lambda(P)$.

Lemma 6.5 *For any $\Lambda > 0$, $K_\Lambda(P)$ has a ground state provided $|P| < \sqrt{2}$.*

Proof. First we recall the following fact: $H_\Lambda(P)$ has a normalized ground state $\Psi_\Lambda(P)$ for $|P| < \sqrt{2}$. As to the proof, see [15, 17, 43]. Corresponding to the decomposition (6.6), one can write

$$\Psi_\Lambda(P) = \Psi_\Lambda^{(0)}(P) \oplus \sum\nolimits_{n\ge 1}^{\oplus} \Psi_\Lambda^{(n)}(P), \tag{6.18}$$

where $\Psi_\Lambda^{(0)}(P) \in \mathfrak{F}_\Lambda$ and $\Psi_\Lambda^{(n)}(P) \in L^2_{\rm sym}(B_\Lambda^{{\rm c}\times n}; \mathfrak{F}_\Lambda)$.

Put $\mathcal{S}_{\Lambda,P} = \{n \in \mathbb{N} \,|\, \Psi_\Lambda^{(n)}(P) \neq 0\}$. Assume $\mathcal{S}_{\Lambda,P}$ is not empty. The we have $1 = \|\Psi_\Lambda(P)\|^2 = \|\Psi_\Lambda^{(0)}(P)\|^2 + \sum_{n\in\mathcal{S}_{\Lambda,P}} \|\Psi_\Lambda^{(n)}(P)\|^2$ and, by (6.7) and Lemma B.1,

$$\begin{aligned} E_\Lambda(P) &= \langle \Psi_\Lambda(P), H_\Lambda(P)\Psi_\Lambda(P)\rangle \\ &\ge \mathcal{E}_\Lambda(P)\|\Psi_\Lambda^{(0)}(P)\|^2 + \sum_{n\ge 1} \big[\mathcal{E}_\Lambda(0)+1\big]\|\Psi_\Lambda^{(n)}(P)\|^2 \\ &= \mathcal{E}_\Lambda(P)\|\Psi_\Lambda^{(0)}(P)\|^2 + \sum_{n\in\mathcal{S}_{\Lambda,P}} \big[\mathcal{E}_\Lambda(0)+1\big]\|\Psi_\Lambda^{(n)}(P)\|^2. \end{aligned} \tag{6.19}$$

By (6.17),

$$\big[\mathcal{E}_\Lambda(0)+1\big] - \mathcal{E}_\Lambda(P) \ge 1 - \frac{P^2}{2} > 0 \tag{6.20}$$

provided $|P| < \sqrt{2}$. Thus, if $|P| < \sqrt{2}$,

$$\text{RHS of (6.19)} > \mathcal{E}_\Lambda(P). \tag{6.21}$$

This contradicts with (6.16). Hence $\mathcal{S}_{\Lambda,P} = \emptyset$ and $\Psi_\Lambda(P) = \Psi_\Lambda^{(0)}(P) \oplus 0 \oplus 0 \oplus \cdots$. Moreover $\Psi_\Lambda^{(0)}(P)$ must be a ground state of $K_\Lambda(P)$. □

Corollary 6.6 *For each $|P| < \sqrt{2}$ and $\Lambda > 0$, the ground state $\Psi_\Lambda^{(0)}(P)$ of $K_\Lambda(P)$ is unique in $\mathfrak{F}_\Lambda$ and can be chosen strictly positive w.r.t. $\mathfrak{F}_{\Lambda,+}$.*

Proof. This immediately follows from the local ergodicity (Proposition 6.3) and Theorem A.6. □

Next we regard $K_\Lambda(P)$ as a self-adjoint operator on a larger subspace $\mathfrak{F}_{\Lambda'}$ for $\Lambda' > \Lambda$. Then $\Psi_\Lambda^{(0)}(P)$ can be regarded as a vector in $\mathfrak{F}_{\Lambda',+}$.

Lemma 6.7 *Let $\Psi_\Lambda^{(0)}(P)$ be the unique ground state of $K_\Lambda(P)$. Then, for each $|P| < \sqrt{2}$ and $\Lambda' > \Lambda$, $\Psi_\Lambda^{(0)}(P)$ is* **not** *the ground state of $K_{\Lambda'}(P)$.*

Proof. Note that $\Psi_\Lambda^{(0)}(P)$ is not strictly positive w.r.t. $\mathfrak{F}_{\Lambda',+}$ anymore (but it is still positive w.r.t. $\mathfrak{F}_{\Lambda',+}$). On the other hand, the ground state of $K_{\Lambda'}(P)$ must be unique and strictly positive by Proposition 6.3 and Theorem A.6. However $\Psi_\Lambda^{(0)}(P)$ is not strictly positive, so that it can not be the ground state of $K_{\Lambda'}(P)$. □

If $\Lambda' > \Lambda$, then one has $Q_\Lambda K_{\Lambda'}(P) Q_\Lambda = Q_\Lambda K_\Lambda(P) Q_\Lambda$. Since $\Psi_\Lambda^{(0)}(P)$ is not the ground state for $K_{\Lambda'}(P)$ by Lemma 6.7, one has

$$\begin{aligned} \mathcal{E}_\Lambda(P) &= \langle \Psi_\Lambda^{(0)}(P), Q_\Lambda K_\Lambda(P) Q_\Lambda \Psi_\Lambda^{(0)}(P)\rangle \\ &= \langle \Psi_\Lambda^{(0)}(P), Q_\Lambda K_{\Lambda'}(P) Q_\Lambda \Psi_\Lambda^{(0)}(P)\rangle \\ &> \mathcal{E}_{\Lambda'}(P). \end{aligned} \tag{6.22}$$

Indeed, suppose

$$\langle \Psi_\Lambda^{(0)}(P), Q_\Lambda K_{\Lambda'}(P) Q_\Lambda \Psi_\Lambda^{(0)}(P)\rangle = \mathcal{E}_{\Lambda'}(P).$$

Then $\Psi_\Lambda^{(0)}(P) = Q_\Lambda \Psi_\Lambda^{(0)}(P)$ must be the ground state of $K_{\Lambda'}(P)$. But this contradicts Lemma 6.7 so that the last inequality in (6.22) holds . Combining (6.22) with Lemma 6.4, one arrives at the desired assertion in Theorem 2.3. □

A Fundamental theoreoms in the self-dual cone analysis

In this section, we will review some preliminary results about the inequalities introduced in §3.1. Almost all of results here are taken from the author's previous work [32, 33].

A.1 Basic tools

Let $\mathfrak{v}$ be a dense subspace of the Hilbert space $\mathfrak{h}$. Set

$$\mathcal{L}(\mathfrak{v}) = \Big\{ A : \text{linear operator on } \mathfrak{h} \text{ s.t. } \mathfrak{v} \subseteq \mathrm{dom}(A), A\mathfrak{v} \subseteq \mathfrak{v}, A^*\mathfrak{v} \subseteq \mathfrak{v} \Big\}. \tag{A.1}$$

Obviously $\mathcal{L}(\mathfrak{v})$ is a linear space and closed under the operator product, that is, if $A, B \in \mathcal{L}(\mathfrak{v})$, then $AB \in \mathcal{L}(\mathfrak{v})$. In this subsection, we always assume every operator in the lemmas belongs to $\mathcal{L}(\mathfrak{v})$. This tacit assumption remove unnecessary complexities on domain problem. For instance, the abnormal case $\mathfrak{p} \cap \mathrm{dom}(A) = \{0\}$ can be avoided automatically. We remark that all operators in the main sections actually satisfy the assumption under a suitable choice of $\mathfrak{v}$.

The following two lemmata are immedate consequences of the definitions.

Lemma A.1 *Suppose that $0 \trianglelefteq A_1 \trianglelefteq B_1$ and $0 \trianglelefteq A_2 \trianglelefteq B_2$ w.r.t $\mathfrak{p}$. Then one has the following.*

(i) $0 \trianglelefteq A_1 A_2$ *w.r.t.* $\mathfrak{p}$. *Moreover if* $A_1, B_1 \in \mathfrak{B}(\mathfrak{h})$, *the set of all bounded operators on* $\mathfrak{h}$, *then* $0 \trianglelefteq A_1 A_2 \trianglelefteq B_1 B_2$ *w.r.t.* $\mathfrak{p}$.

(ii) $0 \trianglelefteq aA_1 + bA_2 \trianglelefteq aB_1 + bB_2$ *w.r.t.* $\mathfrak{p}$, *for all* $a, b \in \mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x \ge 0\}$.

(iii) *Let* A *be positivity preserving:* $0 \trianglelefteq A$ *w.r.t.* $\mathfrak{p}$. *Suppose that* $\mathfrak{p} \cap \mathrm{dom}(A)$ *is dense in* $\mathfrak{p}$. *Then* $0 \trianglelefteq A^*$ *w.r.t.* $\mathfrak{p}$.

Lemma A.2 *Let* $A, B \in \mathfrak{B}(\mathfrak{h})$. *Suppose that* $0 \lhd A$ *and* $0 \trianglelefteq B$ *w.r.t.* $\mathfrak{p}$. *Then we have the following properties.*

(i) $0 \lhd A^*$ *w.r.t.* $\mathfrak{p}$.

(ii) *Suppose that* $\ker B^{\#} = \{0\}$ *with* $a^{\#} = a$ *or* a^*. *Then* $0 \lhd AB$ *and* $0 \lhd BA$ *w.r.t.* $\mathfrak{p}$.

(iii) $0 \lhd aA + bB$ *w.r.t.* $\mathfrak{p}$ *for* $a > 0$ *and* $b \ge 0$.

A.2 Operator monotonicity

Proposition A.3 (Monotonicity) *Let* A *and* B *be positive self-adjoint operators. We assume the following.*

(a) $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ *or* $\mathrm{dom}(A) \supseteq \mathrm{dom}(B)$.

(b) $(A+s)^{-1} \trianglerighteq 0$ *and* $(B+s)^{-1} \trianglerighteq 0$ *w.r.t.* $\mathfrak{p}$ *for all* $s > 0$.

Then the following are equivalent to each other.

(i) $B \trianglerighteq A$ *w.r.t.* $\mathfrak{p}$.

(ii) $(A+s)^{-1} \trianglerighteq (B+s)^{-1}$ *w.r.t.* $\mathfrak{p}$ *for all* $s > 0$.

(iii) $\mathrm{e}^{-tA} \trianglerighteq \mathrm{e}^{-tB}$ *w.r.t.* $\mathfrak{p}$ *for all* $t \ge 0$.

Proof. (i) $\Rightarrow$ (ii): By the assumptions (a) and (b), we see that

$$(A+s)^{-1} - (B+s)^{-1} = (A+s)^{-1}(B-A)(B+s)^{-1} \trianglerighteq 0.$$

(ii) $\Rightarrow$ (iii):

$$\mathrm{e}^{-tA} = \text{s-}\lim_{n\to\infty} (\mathbb{1} + tA/n)^{-n} \trianglerighteq \text{s-}\lim_{n\to\infty} (\mathbb{1} + tB/n)^{-n} = \mathrm{e}^{-tB}.$$

(iii) $\Rightarrow$ (i):

$$A = \text{s-}\lim_{t\downarrow 0} (\mathbb{1} - \mathrm{e}^{-tA})/t \trianglelefteq \text{s-}\lim_{t\downarrow 0} (\mathbb{1} - \mathrm{e}^{-tB})/t = B. \quad \square$$

Proposition A.4 *Let* A *be a positive self-adjoint operator and let* B *be a symmetric operator. Assume the following.*

(i) *B is A-bounded with relative bound $a < 1$, i.e.,* $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ *and* $\|Bx\| \le a\|Ax\| + b\|x\|$ *for all* $x \in \mathrm{dom}(A)$.

(ii) $0 \trianglelefteq \mathrm{e}^{-tA}$ *w.r.t.* $\mathfrak{p}$ *for all* $t \ge 0$.

(iii) $0 \trianglelefteq -B$ *w.r.t.* $\mathfrak{p}$.

Then $\mathrm{e}^{-t(A+B)} \trianglerighteq \mathrm{e}^{-tA} \trianglerighteq 0$ *w.r.t.* $\mathfrak{p}$ *for all* $t \ge 0$.

Proof. See [32]. □

A.3 Beurling-Deny criterion

Let j be the involution given in §3.1. Let A be a linear operator acting in $\mathfrak{h}$. We say that A is *j-real* if $j\mathrm{dom}(A) \subseteq \mathrm{dom}(A)$ and $jAx = Ajx$ for all $x \in \mathrm{dom}(A)$. Set $\mathfrak{h}_{\mathbb{R}} = \{x \in \mathfrak{h} \mid jx = x\}$. Then for any $x \in \mathfrak{h}_{\mathbb{R}}$, we have a unique decomposition $x = x_+ - x_-$ with $x_\pm \in \mathfrak{p}$ and $\langle x_+, x_-\rangle = 0$. Recall the notation $|x|_{\mathfrak{p}} = x_+ + x_-$.

The following theorem is an abstract version of Beurling-Deny criterion [3].

Theorem A.5 (Beurling-Deny criterion) *Let A be a positive self-adjoint operator on $\mathfrak{h}$. Assume that A is j-real. Then the following are equivalent.*

(i) $0 \trianglelefteq \mathrm{e}^{-tA}$ *for all* $t \ge 0$.

(ii) *If* $x \in \mathrm{dom}(A) \cap \mathfrak{h}_{\mathbb{R}}$, *then* $|x|_{\mathfrak{p}} \in \mathrm{dom}(A^{1/2}) \cap \mathfrak{h}_{\mathbb{R}}$ *and* $\langle |x|_{\mathfrak{p}}, A|x|_{\mathfrak{p}}\rangle \le \langle x, Ax\rangle$.

(iii) *If* $x \in \mathrm{dom}(A) \cap \mathfrak{h}_{\mathbb{R}}$, *then* $x_+ \in \mathrm{dom}(A^{1/2}) \cap \mathfrak{h}_{\mathbb{R}}$ *and* $\langle x_+, Ax_+\rangle \le \langle x, Ax\rangle$.

(iv) *If* $x \in \mathrm{dom}(A) \cap \mathfrak{h}_{\mathbb{R}}$, *then* $x_\pm \in \mathrm{dom}(A^{1/2}) \cap \mathfrak{h}_{\mathbb{R}}$ *and* $\langle x_+, Ax_+\rangle + \langle x_-, Ax_-\rangle \le \langle x, Ax\rangle$.

Proof. Proof is a slight modification of [38, Theorem XIII.50]. □

A.4 Perron-Frobenius-Faris theorem

Theorem A.6 (Perron-Frobenius-Faris) *Let A be a positive self-adjoint operator on $\mathfrak{h}$. Suppose that $0 \trianglelefteq \mathrm{e}^{-tA}$ w.r.t. $\mathfrak{p}$ for all $t \ge 0$ and* $\inf\mathrm{spec}(A)$ *is an eigenvalue. Let P_A be the orthogonal projection onto the closed subspace spanned by eigenvectors associated with* $\inf\mathrm{spec}(A)$. *Then the following are equivalent.*

(i) $\dim\mathrm{ran}P_A = 1$ *and* $P_A \triangleright 0$ *w.r.t.* $\mathfrak{p}$.

(ii) $0 \triangleleft (A+s)^{-1}$ *for some* $s > 0$ *w.r.t.* $\mathfrak{p}$.

(iii) *For all* $x, y \in \mathfrak{p}\backslash\{0\}$, *there exists a* $t > 0$ *such that* $0 < \langle x, \mathrm{e}^{-tA}y\rangle$.

(iv) $0 \triangleleft (A+s)^{-1}$ *for all* $s > 0$ *w.r.t.* $\mathfrak{p}$.

(v) $0 \triangleleft \mathrm{e}^{-tA}$ *for all* $t > 0$ *w.r.t.* $\mathfrak{p}$.

Proof. See, e.g., [8, 32, 38]. □

B An energy inequality

Lemma B.1 *For all $P \in \mathbb{R}^3$ and $0 < \Lambda < \infty$, one has*

$$\mathcal{E}_\Lambda(0) \le \mathcal{E}_\Lambda(P). \tag{B.2}$$

Sketch of proof. Since we need a special self-dual cone different from $\mathfrak{F}_+$, we separate the proof of (B.2) from the main body.

In this appendix, we switch our representation space to the Q-space or the Schrödinger representation. In this representation, the local Fock space $\mathfrak{F}_\Lambda$ can be identified with $L^2(Q_\Lambda) = L^2(Q_\Lambda, \mathrm{d}\mu_\Lambda)$, where μ_Λ is a Gaussian measure, see [39] for details. Let

$$L^2(Q_\Lambda)_+ = \{F \in L^2(Q_\Lambda) \,|\, F \ge 0 \,\text{a.e.}\}. \tag{B.3}$$

Clearly $L^2(Q_\Lambda)_+$ is a self-dual cone in $L^2(Q_\Lambda)$. The conjugation C in the one particle space is given by $(Cf)(k) = \overline{f}(-k)$. Then, by a general theorem [39, Theorem I. 12 and its remark], one sees

$$\mathrm{e}^{\mathrm{i}a\cdot P_{\mathrm{f},\Lambda}} \trianglerighteq 0, \quad \mathrm{e}^{-tN_{\mathrm{f},\Lambda}} \trianglerighteq 0, \quad \mathrm{e}^{tV_\Lambda} \trianglerighteq 0 \tag{B.4}$$

w.r.t. $L^2(Q_\Lambda)_+$, as operators in the Q-space. Hence, following Gross [21], one has

$$\left|\mathrm{e}^{-t(P-P_{\mathrm{f},\Lambda})^2} F\right| \le \mathrm{e}^{-tP_{\mathrm{f},\Lambda}^2} |F| \quad \text{a.e.} \tag{B.5}$$

for each $F \in L^2(Q_\Lambda)$. This implies $|\mathrm{e}^{-tL(P)}F| \le \mathrm{e}^{-tL(0)}|F|$ a.e.. By the Trotter-Kato formula, one obtains $|\mathrm{e}^{-tK_\Lambda(P)}F| \le \mathrm{e}^{-tK_\Lambda(0)}|F|$ a.e.. From this, it follows $\langle F, \mathrm{e}^{-tK_\Lambda(P)}F\rangle \le \langle |F|, \mathrm{e}^{-tK_\Lambda(0)}|F|\rangle$ for all $F \in L^2(Q_\Lambda)$. Now we arrive at the desired result (B.2). □

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